# Adventures of the coupled Yang–Mills oscillators: II. YM–Higgs quantum mechanics

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**Abstract**

We continue our study of the quantum mechanical motion in the $x^2 y^2$ potentials for $n = 2, 3$, which arise in the spatially homogeneous limit of the Yang–Mills (YM) equations. In the present paper, we develop a new approach to the calculation of the partition function $Z(t)$ beyond the Thomas–Fermi (TF) approximation by adding a harmonic (Higgs) potential and taking the limit $v \to 0$, where $v$ is the vacuum expectation value of the Higgs field. Using the Wigner–Kirkwood method to calculate higher-order corrections in $\hbar$, we show that the limit $v \to 0$ leads to power-like singularities of the type $v^{-n}$, which reflect the possibility of escape of the particle along the channels in the classical limit. We show how these singularities can be eliminated by taking into account the quantum fluctuations dictated by the form of the potential.

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## 1. Introduction

Here we continue our study of the quantum mechanical motion in the $x^2 y^2$ potentials of phase space dimensions $2n$ with $n = 2, 3$, which arise in the spatially homogeneous limit of the Yang–Mills (YM) equations. As is well known [1] (see [2] for a review), these systems exhibit a rich chaotic behaviour despite their extreme simplicity. In particular, the $n = 2$ model, the central object of this and also our previous investigation (see [3]—we will henceforth refer to this work as I), has been widely studied.

Classically, this model possesses a logarithmically divergent volume of energetically accessible phase space $\Gamma_E$ [2, 4],[4] but its quantum mechanical version (YM quantum mechanics, YMQM) has a discrete spectrum [6, 7]. Physically, the explanation is obvious: quantum fluctuations, e.g. zero-point fluctuations, forbid that the trajectory escapes along the

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[4] This is in violation of Weil's famous theorem [5], which states that the average energy-level density $dN/dE$ is asymptotically proportional to $\Gamma_E$.

$x$- or $y$-axis where the potential energy vanishes. The system is thus confined to a finite volume, and this implies the discreteness of the energy levels. Classically this escape is always possible without increasing energy. As we shall see below, these classically allowed configurations result in singularities of the quasiclassical partition function.

In I we calculated the higher-order quantum corrections to the partition function (heat kernel) $Z(t)$ for the $x^2y^2$ potential using the approximation [8, 9] based on the adiabatic separation of the motion in $x$ and $y$ in the hyperbolic channels of the equipotential surface $xy = \text{const}$. The main assumption of this method is that the final results of the calculations do not depend on the artificial boundary $Q$ dividing the central region $x, y \in [-Q, Q]$ from the channels $x, y \in [Q, \infty]$. We showed in I that this assumption, after improvement of the quantum mechanical treatment of the motion in the channels, is correct not only for the Thomas–Fermi (TF) term but also for the leading (in powers of $tQ^4 \gg 1$) higher-order quantum corrections, and we derived $Q$-independent asymptotic series in the parameter $\lambda^2 = g^2\hbar^4t^3$ for contribution of each region to $Z(t)$.

In the present paper, as in I, we explore the properties of the $x^2y^2$ model beyond the TF approximation, but we pursue a different approach. We calculate the higher-order corrections to $Z(t)$ as in [10] by starting from the Yang–Mills–Higgs quantum mechanics (YMHQM) and taking the limit $v \to 0$, where $v$ is the vacuum expectation value of the Higgs field. In the $n = 2, 3$ $x^2y^2$ models, $v$ determines the strength of the harmonic potential

$$V(x, y) = \tfrac{1}{2}v^2(x^2 + y^2) \qquad (n = 2),$$
$$V(x, y, z) = \tfrac{1}{2}v^2(x^2 + y^2 + z^2) \qquad (n = 3).$$

Due to the above-mentioned logarithmic infinity of the classical phase space volume of the $x^2y^2$ model it is impossible to completely disentangle the nonlinear coupled oscillators from the harmonic oscillations generated by the Higgs potential leading to a term in $Z(t)$ proportional to $\ln v$. In the $n = 3$ case, which has a finite phase space volume at fixed energy, this method yields an expression for $Z(t)$ that coincides in the limit $v \to 0$ with the one obtained by the adiabatic separation method [9]. This is due to the negligible time spent by the classical trajectory in the depth of the hyperbolic channels [9]. Higher-order corrections change this situation essentially, as we shall see below.

Here we use the approach of [10] with the limiting procedure $v \to 0$ for the calculation of $Z(t)$ beyond the TF approximation by applying the Wigner–Kirkwood (WK) method [11–13] (see [14] for a review of the WK method). The higher-order corrections $\mathcal{O}(\hbar^k)$ in the WK approach lead to a new phenomenon in the limit $v \to 0$: for $k \geqslant 2$, they yield power-like singularities of the form $v^{-k}$. These singularities are not cancelled at a given power $k$ as one might expect. The situation is completely different when one includes the quantum fluctuations in the channels of the $x^2y^2$ potential to all orders. These generate a confining potential, which does not disappear in the limit $v \to 0$, closes the flat direction and eliminates the mentioned singularities, which are essentially classical. Taking these fluctuations into account, we are able to compare the expression for $Z(t)$ obtained by our method with the result for $Z(t)$ obtained by the method of [8, 9] in the TF approximation.

Concerning the $n = 3$ case with its finite phase space volume, the singularities appear also at the higher-order corrections in contrast to the TF approximation and again are eliminated by the quantum fluctuations corresponding to the specific quartic form of the potential characteristic of the YM quantum mechanics[5].

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[5] In the corresponding supersymmetric Yang–Mills quantum mechanical system, the wavefunction is not confined due to the cancellations between bosonic and fermionic degrees of freedom, and a continuous spectrum coexists with a discrete one [15].

Finally, we develop a novel approximation scheme, which relies on a resummation of certain terms in the WK expansion and thus introduces a nonvanishing Higgs potential, which avoids the divergences of the $1/v^k$ approximation. This approach is motivated by the need to take the quantum fluctuations inside the hyperbolic channels into account even in the lowest order approximation. We show that the new approach reproduces the TF result obtained with the method of [8, 9] without requiring an artificial subdivision of the phase space into different regions. A modified WK expansion can be derived to systematically improve on this result.

In the next two sections we present the YMH system and the WK method of calculating $Z(t)$ beyond the TF approximation. In section 4 we analyse the singularity structure of the higher-order terms in the WK expansion in the limit of vanishing Higgs potential. In section 5 we show how these singularities can be avoided by a rearrangement of the WK series. Finally, we briefly consider the three-dimensional YM–Higgs system.

## 2. Yang–Mills–Higgs classical and quantum mechanics: the Thomas–Fermi approximation

There are several mechanisms that can suppress and even eliminate the classical chaos of the YM system (see [2]). One is the Higgs mechanism [16]. For spatially homogeneous fields (long wavelength limit of YM system), if only the interaction of the gauge fields with the Higgs vacuum is considered, the classical Hamiltonian for $n = 2$ is given by the expression

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{g^2}{2}x^2 y^2 + \frac{v^2}{2}(x^2 + y^2), \tag{1}$$

where $v = \langle\phi\rangle$ is the vacuum expectation value of the Higgs field $\phi$. It is known [16] that there is a classical transition from chaos to regular motion as $v$ gets large enough. More precisely, the chaos disappears when $g^2 v^4/E > 0.6$ [16], where $E = H$ is the energy. The analogous transition in the adjacent energy-level spacing distribution was predicted [17] and established in several papers [18]. The quantized counterpart of (1) is

$$\hat{H} = -\frac{\hbar^2}{2}\nabla_{x,y}^2 + \frac{g^2}{2}x^2 y^2 + \frac{v^2}{2}(x^2 + y^2). \tag{2}$$

As in I, we measure all quantities in units of the energy $E$ with dimensions $[H] = 1$, $[t] = -1$, $[x]$, $[y] = 1/4$, $[g] = 0$, $[v] = 1/4$, $[\hbar] = 3/4$.

It is obvious that the operator (2) has a discrete spectrum as it has for $v = 0$. The TF approximation to the heat kernel or partition function $Z(t) = \text{Tr}[\exp(-t\hat{H})]$ is the standard lowest order semiclassical approximation valid for small $\hbar t^{3/4} \ll 1$. It is obtained by substituting the classical Hamiltonian for its quantum counterpart and replacing the trace of the heat kernel by the integral over the phase space volume normalized by $(2\pi\hbar)^{-n}$, where $2n$ is the phase space dimension. In other words, the TF approximation takes into account only the discreteness of the quantum mechanical phase space, but considers momenta and coordinates (in our case, the field amplitudes $x$ and $y$) as commuting variables. This method was used in numerous papers (see, e.g. [8–10]). For the calculation of the energy-level density $\rho(E) = dN(E)/dE$ at asymptotic energies, the TF approximation is a consistent approach since, as we shall see below, all corrections to the TF term are structures with factors $\hbar^k t^\ell$ with $k, \ell$ being positive integers. For the asymptotic energy-level density $\rho(E)$ or $N(E)$ these corrections are negligible according the Karamata–Tauberian theorems [6, 7] relating the most singular part of $Z(t)$ to the asymptotic level density, $N(E) = \int dE\,\rho(E) = \mathcal{L}^{-1}[Z(t)/t]$, where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform.

In [10] $Z(t)$ and $N(E)$ were calculated for the Hamiltonian (1). We give the precise expression for $Z(t)$ of the YMHQM system in the TF approximation

$$Z(t) = \frac{1}{\sqrt{2\pi}\, g\hbar^2 t^{3/2}} \exp\left(\frac{tv^4}{4g^2}\right) K_0\left(\frac{tv^4}{4g^2}\right), \tag{3}$$

where $K_0(z)$ is the modified Bessel function of the third kind. For the most interesting limit $v \to 0$ we get

$$Z(t) \to \frac{1}{\sqrt{2\pi}\, g\hbar^2 t^{3/2}}\left(\ln\frac{8g^2}{tv^4} - C\right), \tag{4}$$

where $C$ is the Euler constant. The impossibility of disentangling the coupled oscillators from the uncoupled ones is expressed by the logarithmic divergence of the phase space volume for $n = 2$ as we already mentioned in the introduction. Below we compare (4) with the corresponding expression for $Z(t)$ obtained in [8, 9] for the pure $x^2y^2$ model. Because we shall often encounter the pre-factor appearing in (3) and (4) in the following, we introduce the special symbol $K$ for it:

$$K = (2\pi g^2 \hbar^4 t^3)^{-1/2} \equiv (2\pi\lambda^2)^{-1/2}. \tag{5}$$

## 3. Beyond the TF approximation: the Kirkwood–Wigner expansion

In the present paper, unlike in I, we apply the WK expansion in all of phase space, avoiding the division of the phase space into a central region and hyperbolic channels. As we will see below, this poses no problems as long as $v \neq 0$. However, singularities appear in the limit $v \to 0$, unless the WK expansion is modified to include quantum fluctuations in a nonperturbative way.

Since we described the WK method in detail in I, we only give a very brief outline here. We start from equation (I-12), a set of recurrent differential equations for the kernels $W_k$ of the partition function $Z_k(t)$ at the $k$th order in $\hbar$:

$$Z_k(t) = \frac{\hbar^k}{(2\pi\hbar)^2} \int d\Gamma\, W_k(\vec{r}, \vec{p}; t)\, e^{-tH}, \tag{6}$$

where $d\Gamma = dx\, dy\, dp_x\, dp_y$ and the classical Hamiltonian $H$ is given by (1). We begin with the partition function at second order in $\hbar$ ($k = 2$). Integrating over $p_x$ and $p_y$ and making use of the symmetry of the Hamiltonian (1) with respect to the interchange $x \leftrightarrow y$, we obtain

$$\int d\Gamma\, W_2\, e^{-tH} = \frac{\pi t}{3}\left[\left(-g^2 + \frac{tv^4}{2}\right) I_{10} + \frac{tg^4}{2} I_{21} - v^2 I_{00} + tg^2 v^2 I_{11}\right], \tag{7}$$

where we have used the notation (for $m \geqslant n$):[6]

$$I_{mn} = 4 \int_0^\infty dx \int_0^\infty dy\, x^{2m} y^{2n} \exp\left[-\frac{t}{2}(v^2(x^2 + y^2) + g^2 x^2 y^2)\right]. \tag{8}$$

Note that the factors $\hbar^2$ from the KW expansion and from the normalization of the phase space volume element have cancelled, making $Z_2$ independent of $\hbar$. Integrating over $y$ and introducing the new variable $u = g^2 x^2/v^2$, we obtain

$$I_{mn} = \frac{\sqrt{2\pi}(2n-1)!!(v^2)^{m-n}}{t^{n+\frac{1}{2}} g^{2m+1}} \int_0^\infty du\, u^{m-\frac{1}{2}}(1+u)^{-n-\frac{1}{2}}\, e^{-uz} \tag{9}$$

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[6] Note that the case $m = n$ needs to be calculated separately from the case $m > n$.

with $z = tv^4/2g^2$. The integral over $u$ is related to the Whittaker function $W_{\lambda,\mu}(z)$ (see [19], equation (9.222.1)):

$$I_{mn} = \frac{\sqrt{2\pi}(2n-1)!!(v^2)^{m-n}}{t^{n+\frac{1}{2}}g^{2m+1}} \, e^{z/2} z^{-\frac{m-n+1}{2}} \Gamma\left(\frac{2m+1}{2}\right) W_{-\frac{m+n}{2},\frac{m-n}{2}}(z). \quad (10)$$

For $m = n$ the Whittaker function has only a logarithmic divergence at $z = 0$. For $m - n \geqslant 1$ power-like singularities appear. For completeness we give $Z_2(t)$ explicitly

$$Z_2(t) = \frac{tv^2}{12\sqrt{2\pi}\,gt^{1/2}}\left[2\left(-1+\frac{tv^4}{2g^2}\right)z^{-1}\Gamma\left(\frac{3}{2}\right)W_{-\frac{1}{2},\frac{1}{2}}(z) + z^{-1}\Gamma\left(\frac{5}{2}\right)W_{-\frac{3}{2},\frac{1}{2}}(z)\right.$$
$$\left. - 2z^{-1/2}\Gamma\left(\frac{1}{2}\right)W_{0,0}(z) + 2z^{-1/2}\Gamma\left(\frac{3}{2}\right)W_{-1,0}(z)\right]. \quad (11)$$

As is easily seen from (11), $Z_2(t)$ has a singularity of the form $v^{-2}$ at $v = 0$. Using the limit of the Whittaker function $W_{\lambda,\mu}(z)$ for small $z$, we find

$$Z_2(t) \rightarrow -K\frac{\hbar^2 g^2 t^{3/2}}{6(tv^4)^{1/2}} \qquad (v \rightarrow 0). \quad (12)$$

For completeness we give here $Z_2$ from the KW method in the limit $g = 0$, i.e. for two free harmonic oscillators using the asymptotic form of the Whittaker function

$$Z_2(t) = -\tfrac{1}{12}. \quad (13)$$

Together with the expression for the TF term [10] we have

$$Z_0(t) + Z_2(t) = \frac{1}{\hbar^2 v^2 t^2}\left(1 - \frac{1}{12}\hbar^2 v^2 t^2\right), \quad (14)$$

which are the first two terms in the Taylor expansion of the exact partition function for the two-dimensional harmonic oscillator at small $\hbar vt$:

$$Z(t) = \left[2\sinh\tfrac{1}{2}\hbar vt\right]^{-2}. \quad (15)$$

## 4. Higher-order corrections and the limit $v \rightarrow 0$ for YMHQM

At higher order $\hbar^k$ ($k \geqslant 2$) the mathematical structure of terms $W_k$ and $Z_k$ changes essentially. In expression (8) higher powers $m$ and $n$ appear and the difference between them increases $(m - n \geqslant 1)$, causing the second index of the Whittaker function to exceed $\mu = 1/2$. As a result, in the limit $z = tv^4/2g^2 \rightarrow 0$ the finite sum of the power-like singular terms of $W_{\lambda,\mu}(z)$ begins to play a crucial role.

A systematic analysis of the higher-order corrections using *Mathematica* leads to the conclusion that there is a correlation between the power of $\hbar$ (for $k \geqslant 2$) and the most singular terms in (10): $m - n = k/2$ (due to the symmetry against exchange $x \leftrightarrow y$ it is always possible to put $m > n$). The case $m = n$ requires special consideration and, as found, leads only to logarithmic singularities.

It is easy to see that the next, less singular terms correspond to the case $m - n = (k/2) - 2\ell$ ($\ell = 1, 2, \ldots, \ell < k/4$). For the general expression of $Z_k^{(m,n)}(t)$ we need to determine the powers of $g^2$ and $t$. For $g^2$ it is simply $g^{2m}$. To determine the power of $t$ we note that for each $k$ there are always terms without factors of $p_x$ and $p_y$ and minimal power of $t$ at given $k$. For such terms the factor is $t^{2m-n-1}$. For the terms containing factors of $p_x$ and $p_y$ the power of $t$ does not change after integration over the momenta. For the less singular terms with $m - n = \frac{1}{2}k - 2\ell$ the corresponding factor is $t^{2m-n-1-3\ell}$.

Now we are in a position to write the general expression for the most singular terms contributing to the partition function at the order of $\hbar^k$ ($k = 2, 4, 6, \ldots$). After integration over $p_x$, $p_y$, $x$ and $y$, keeping only terms having $m - n = k/2$ ($k/2 \leqslant m \leqslant k, 0 \leqslant n \leqslant k/2$) we obtain

$$Z_k^{(m,n)}(t) = K \left( \frac{4g^4 \hbar^4 t^3}{v^4 t} \right)^{k/4} \Gamma \left( \frac{k}{2} \right) (2m - k - 1)!!. \tag{16}$$

We see that there are strong singularities at $v = 0$, and further analysis shows that these are not cancelled by the summation of all most singular terms at a given $k$. For the less singular terms with $m - n = k/2 - 2\ell$ with $\ell < k/4$ we have

$$Z_k^{(m,n,\ell)}(t) = K \left( \frac{4g^4 \hbar^4 t^3}{v^4 t} \right)^{k/4} \left( \frac{v^4 t}{4g^4} \right)^{\ell} \Gamma \left( \frac{k}{2} - \ell \right) (2m - k - 1)!!. \tag{17}$$

Due to the appearance of $(m - n)$ in the argument of the Gamma function in (17), the case with $m = n$ must be considered separately. It is obvious that these terms have no power-like divergences, but only logarithmic singularities like the TF term. We obtain

$$Z_k^{(m=n)}(t) = K (g^2 \hbar^4 t^3)^{k/4} \left[ \ln \frac{2g^2}{v^4 t} - 2C - \psi \left( m + \frac{1}{2} \right) \right], \tag{18}$$

where $\psi(x)$ is the logarithmic derivative of the Gamma function. Let us briefly discuss these results. The power-like singularities in the KW approach for $k \geqslant 2$ are related to the possibility of escaping classically along the axis $x = 0$ or $y = 0$, where, in the limit $v \to 0$, the potential energy vanishes. Non-zero $v$ forbids such escape to infinity. These singularities affect any classically calculated distribution function, in particular, the heat kernel $Z(t)$. They also show that the trajectories lie deep inside the channels most of the time. Quantum mechanical fluctuations forbid any escapes along the axes.

The $v^{-k}$ singularities have more resemblance with the infrared singularities; they are related to the behaviour at long distances and different from the usual ultraviolet divergences connected with the asymptotic expansion in powers of $h^k$. The absence of power-like singularities for $m = n$ is easily explained: for $m = n$ the configurations dominate along the diagonals ($|x| = |y|$), whereas for $m \gg n$ the configurations populate the channels and have a trend to escape if quantum fluctuations do not forbid this. It is thus clear that we have to take into account the effect of the quantum fluctuations on the motion inside the channels, which the perturbative expansion of the WK method fails to do.

## 5. Quantum fluctuations and power-like singularities

In this section we will attempt to include quantum fluctuations created by the form of $x^2 y^2$ potential in a framework, which does not rely directly on the adiabatic separation of the motion inside the hyperbolic channels as was elaborated in detail in I. Let us consider the motion along the $x$-axis. The heat kernel for the $x^2 y^2$ potential generates a mean spread in $y$ at the position $x$ of the order of $\delta y \sim (g^2 x^2 (t/2)^{-1/2})$. Quantum mechanics dictates that the spread of the conjugate momentum $p_y$ is at least $\delta p_y \geqslant \hbar g |x| (t/2)^{1/2}$. Analogous relations hold between $y$ and $p_x$: $\delta p_x \geqslant \hbar g |y| (t/2)^{1/2}$ for the motion along the $y$-axis. We now propose a modification of the WK formalism, which incorporates these relationships into the generating functional for the expansion in powers of $\hbar$.

To achieve this, we resum the term $-\frac{1}{2} \hbar^2 t (\Delta V)$ in the WK operator (I-10) to all orders by writing

$$W(\vec{r}, \vec{p}; t) = \tilde{W}(\vec{r}, \vec{p}; t) \exp \left( -\frac{\hbar^2}{4} t^2 \Delta V \right). \tag{19}$$

Inserting this definition into the differential equation (I-10) for the WK kernel $W$ yields an equation for the phase space function $\tilde{W}$:

$$\frac{\partial \tilde{W}}{\partial t} = \frac{\hbar^2}{2}\left[\Delta + t^2(\nabla V)^2 - \frac{2\mathrm{i}t}{\hbar}(\vec{p}\cdot\nabla V) - \frac{\hbar^2}{4}t^2(\Delta\Delta V) + \frac{2}{\hbar}(\mathrm{i}\vec{p} - \hbar t\nabla V)\cdot\nabla\right.$$
$$\left. - \frac{\hbar^2}{2}t^2\nabla(\Delta V)\cdot\nabla - \frac{\hbar}{2}t^2(\mathrm{i}\vec{p} - \hbar t\nabla V)\cdot\nabla(\Delta V)\right]\tilde{W}. \tag{20}$$

The term in the exponent of (19):

$$-\frac{\hbar^2}{4}t^2\Delta V = -\frac{\hbar^2}{4}g^2t^2(x^2 + y^2) \tag{21}$$

acts like a Higgs potential with

$$v_{\mathrm{eff}}^2 = \hbar^2 g^2 t/2. \tag{22}$$

What is unusual about this term is that the effective potential itself is time dependent. The connection to the argument given at the beginning of this section becomes evident, when one recognizes that the exponent represents the lower bound associated with the kinetic energy demanded by the uncertainty relation:

$$\frac{1}{2}\left(\delta p_x^2 + \delta p_y^2\right) \geqslant \frac{\hbar^2}{4}t\Delta V. \tag{23}$$

It is straightforward to derive a recursion relation analogous to (I-12) for the coefficients of the expansion of $\tilde{W}$ in powers of $\hbar$:[7]

$$\frac{\partial \tilde{W}_k}{\partial t} = \mathrm{i}\vec{p}\cdot[\nabla - t(\nabla V)]\tilde{W}_{k-1} + \frac{1}{2}[\Delta + t^2(\nabla V)^2 - 2t\nabla V\cdot\nabla]\tilde{W}_{k-2}$$
$$- \left[\frac{\mathrm{i}t^2}{4}\vec{p}\cdot\nabla(\Delta V)\right]\tilde{W}_{k-3}$$
$$+ \left[\frac{t^3}{4}\nabla V\cdot\nabla(\Delta V) - \frac{t^2}{4}\nabla(\Delta V)\cdot\nabla + \frac{t^2}{8}\Delta\Delta V\right]\tilde{W}_{k-4}. \tag{24}$$

Before we apply this trick to the quartic YM oscillator, it is useful to briefly explore how it works for the harmonic oscillator. It is easy to see that (24) yields the correct expansion for the partition function for the potential $V = \frac{1}{2}v^2x^2$. Indeed, by calculating $\tilde{W}_0$, $\tilde{W}_2$, $\tilde{W}_4$ and integrating over $p$ and $x$, we obtain

$$\tilde{Z}_0(t) = \frac{1}{\hbar vt}\,\mathrm{e}^{-(\hbar vt)^2/4},$$
$$\tilde{Z}_2(t) = \frac{5}{24}\hbar vt\,\mathrm{e}^{-(\hbar vt)^2/4}, \tag{25}$$
$$\tilde{Z}_4(t) = \frac{127}{5760}(\hbar vt)^3\,\mathrm{e}^{-(\hbar vt)^2/4},$$

giving the expansion in powers of $(\hbar vt)$ (up to $\hbar^4$) of the exact partition function of the harmonic oscillator, $Z(t) = (2\sinh\hbar vt/2)^{-1}$. Note that, in this case, $\tilde{Z}_k$ contributes to all powers in the $\hbar$ expansion of $Z(t)$ beginning with $\hbar^{k-1}$.

Now consider the nonlinear potential of the YM–Higgs Hamiltonian (2), for which $\Delta V = g^2(x^2 + y^2) + 2v^2$. The lowest order term of the partition function for the modified WK expansion is easily obtained from (3) by substituting $v^2 \to v^2 + v_{\mathrm{eff}}^2$:

$$\tilde{Z}_0 = K\exp\left(\frac{t}{4g^2}\left(v^2 + v_{\mathrm{eff}}^2\right)^2 - \frac{\hbar^2}{2}v^2t^2\right)K_0\left(\frac{t}{4g^2}\left(v^2 + v_{\mathrm{eff}}^2\right)^2\right). \tag{26}$$

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[7] Note that the expansion of $\tilde{W}$ in powers of $\hbar$ does not yield an expansion of $Z(t)$ strictly in powers of $\hbar$ because of the nonpolynomial factor containing $\hbar$ in (19).

The $v \to 0$ limit of this expression is obviously nonsingular, allowing us to go to the pure Yang–Mills limit:

$$\tilde{Z}_0^{(v=0)} = K \exp\left(\frac{\hbar^4}{16} g^2 t^3\right) K_0\left(\frac{\hbar^4}{16} g^2 t^3\right) \approx K \left[\ln \frac{1}{g^2 \hbar^4 t^3} + 5 \ln 2 - C\right], \tag{27}$$

which coincides (with logarithmic precision) with the result obtained in [8, 9] for $Z(t)$. Similarly, the second-order term (in the limit $v \to 0$) is

$$\tilde{Z}_2^{(v=0)} = \frac{g^2 t}{24\pi} (4\tilde{I}_{10} + g^2 t \tilde{I}_{21}) \tag{28}$$

with

$$\tilde{I}_{mn} = 4 \int_0^\infty dx \int_0^\infty dy\, x^{2m} y^{2n} \exp\left(-\frac{1}{2} g^2 x^2 y^2 t - \frac{\hbar^2}{4} g^2 (x^2 + y^2) t^2\right). \tag{29}$$

These integrals correspond to those defined in (8) with the substitution $v^2 \to v_{\text{eff}}^2$. The exact analytical expression for these integrals in terms of Whittaker functions was given in (10). In the following we retain only the first term in the (finite) power series expansion of the Whittaker function; later we shall correct for this simplification. Using the condition $z = g^2 \hbar^4 t^3/8 \equiv \lambda^2/8 \ll 1$, allowing us to neglect terms involving $\ln \lambda^2$ compared with terms of the form $\lambda^{-2}$, we obtain

$$\tilde{I}_{10} = \frac{\sqrt{2\pi}}{(g^2 t)^{1/2}} \frac{4}{(\hbar g t)^2}, \qquad \tilde{I}_{21} = \frac{\sqrt{2\pi}}{(g^2 t)^{3/2}} \frac{4}{(\hbar g t)^2}, \tag{30}$$

yielding the result

$$\tilde{Z}_2(t) = \tfrac{5}{3} K. \tag{31}$$

For $\tilde{Z}_4$ we retain only terms with $m - n = 2$ as the most singular ones in the limit $v \to 0$, based on the usual arguments. This gives

$$\tilde{Z}_4(t) = \frac{(\hbar g t)^4}{2\pi \hbar^2 t} \left[\frac{1}{30} \tilde{I}_{20} + \frac{g^2 t}{180} \tilde{I}_{31} + \frac{(g^2 t)^2}{576} \tilde{I}_{42}\right]. \tag{32}$$

For the needed integrals we obtain in the same approximation ($\lambda^2 \ll 1$)

$$\tilde{I}_{20} = \frac{\sqrt{2\pi}}{(g^2 t)^{1/2}} \frac{16}{(\hbar g t)^4}, \qquad \tilde{I}_{31} = \frac{\sqrt{2\pi}}{(g^2 t)^{3/2}} \frac{16}{(\hbar g t)^4}, \qquad \tilde{I}_{42} = \frac{\sqrt{2\pi}}{(g^2 t)^{5/2}} \frac{48}{(\hbar g t)^4}. \tag{33}$$

Substituting (33) into (32) we obtain

$$\tilde{Z}_4(t) = \frac{127}{180} K. \tag{34}$$

Collecting all results, we have

$$\tilde{Z}_{0+2+4}(t) = K \left[\ln \frac{1}{g^2 \hbar^4 t^3} + 5 \ln 2 - C + \frac{427}{180}\right]. \tag{35}$$

We improve upon the approximation made above to the Whittaker function and consider the contribution of all singular terms in the asymptotic expansion of $W_{\kappa,\mu}(z)$ for small $z$:

$$\sum_{p=0}^{m-n-1} \Gamma(m - n - p) \Gamma\left(n + \frac{1}{2} + p\right) \left(-\frac{\lambda^2}{8}\right)^p, \tag{36}$$

where again $\lambda^2 = g^2 \hbar^4 t^3$. The complete expression for the contribution to $\tilde{Z}_k$ ($k = 2, 4, 6, \ldots$) from the most singular terms is

$$\tilde{Z}_k^{(m,n)} = K \frac{2^k (2n - 1)!!}{\Gamma\left(n + \frac{1}{2}\right)} \sum_{p=0}^{\frac{1}{2}k - 1} \Gamma\left(\frac{1}{2} k - p\right) \Gamma\left(n + \frac{1}{2} + p\right) \left(-\frac{\lambda^2}{8}\right)^p. \tag{37}$$

For the less singular terms $\left(m - n = \frac{1}{2}k - 2\ell, \ \ell < \frac{1}{4}k\right)$ we get

$$\tilde{Z}_k^{(m,n,\ell)} = K\lambda^{2\ell} \frac{2^{k-4\ell}(2n-1)!!}{\Gamma\left(n + \frac{1}{2}\right)} \sum_{p=0}^{\frac{1}{2}k - 2\ell - 1} \Gamma\left(\frac{1}{2}k - 2\ell - p\right) \Gamma\left(n + \frac{1}{2} + p\right) \left(-\frac{\lambda^2}{8}\right)^p, \quad (38)$$

and for the logarithmic term $(m = n)$ we obtain

$$\tilde{Z}_k^{(m=n)} = K\lambda^{\frac{1}{2}k} \left[ -\ln\lambda^2 + 3\ln 2 - 2C - \psi\left(m + \frac{1}{2}\right) \right]. \quad (39)$$

It is clear that these contributions again generate an asymptotic series in the small parameter $\lambda^2 = g^2\hbar^4 t^3$ of the following form:

$$Z(t) = K\left[ -\ln\lambda^2 + 5\ln 2 - C + \sum_{k=2,4\ldots} 2^k \sum_{n=0}^{k/2} a_n^{(k/2)} \frac{2^k(2n-1)!!}{\Gamma\left(n + \frac{1}{2}\right)} \right.$$

$$\left. \times \sum_{p=0}^{\frac{1}{2}k - 1} \Gamma\left(\frac{1}{2}k - p\right) \Gamma\left(n + \frac{1}{2} + p\right) \left(-\frac{\lambda^2}{8}\right)^p \right], \quad (40)$$

where the $a_n^{(k/2)} \left(n = 0, 1, \ldots, \frac{k}{2}\right)$ are the coefficients of the structures $(g^2 t)^n \tilde{I}_{mn} \left(m - n = \frac{k}{2}\right)$ in the expression $\int d\Gamma \, \tilde{W}_k \, e^{-Ht}$ $(k = 2, 4, 6, \ldots)$, analogous to the coefficients $a_m^{(n)}$ $(m = 0, 1, \ldots, 3n)$ in (I-71). For $k = 4$, e.g., these numbers are

$$a_0^{(2)} = \frac{1}{30}, \qquad a_1^{(2)} = \frac{1}{180}, \qquad a_2^{(2)} = \frac{1}{576}. \quad (41)$$

We note that this asymptotic series closely resembles the one derived in I using a completely different approach, splitting the $x - y$ plane into two integration regions and treating the quantum fluctuations exactly in the region containing the hyperbolic channels. The leading logarithmic term is identical in both cases, but it is not clear that the constant coincides. The expansion parameter $\lambda^2$ is the same for both series. Unfortunately, our inability to find a simple general algorithm for the coefficients, $a_n^{(k/2)}$ here and $a_m^{(n)}$ in I, has prevented us to compare the two results in detail. The less singular terms also lead, after summation over $k$, to an asymptotic series in the parameter $\lambda^2$. The same is true for the terms $Z_k^{(m,m)}$ with $m = n$.

At the end of section 4 we commented on the analogy between the limit $v \to 0$ and the infrared problem. We now can make this analogy more precise. The infrared limit corresponds to the behaviour of the system at large distances or deep inside the channels, in the terminology of the paper I. When we compare the limiting forms for $v \to 0$ of the quantities $I_{mn}$ derived here with expression (I-25) for the analogous quantities derived for large distances $(t Q^4 \gg 1)$ in I, we obtain the correspondence

$$Q^{2(m-n)} \sim \frac{m - n}{\Gamma\left(n + \frac{1}{2}\right)} \frac{1}{(tv^2)^{m-n}}. \quad (42)$$

Evidently the limit $v \to 0$ corresponds to the limit $Q \to \infty$, supporting our claim that it represents the infrared limit.

## 6. The three-dimensional YMHQM

Finally, we briefly consider the $n = 3$ case of the YMHQM model. The Hamiltonian for $n = 3$ YMH classical mechanics is

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{g^2}{2}(x^2 y^2 + y^2 z^2 + z^2 x^2). \quad (43)$$

In its quantum counterpart, $\vec{p}^2$ is replaced with $-\hbar^2 \nabla^2$. As we know the contribution to $Z(t)$ from the channels is negligible in the TF approximation [9] if we apply the condition $tQ^4 \gg 1$. This is due to the fact that deep in a channel, e.g. along the $x$-axis, the energetically accessible phase space volume gets pinched as $x^{-2}$ and not as $x^{-1}$ as it was for the $n = 2$ Hamiltonian. This implies that the limit $v \to 0$ is smooth and the expressions for $Z(t)$ from [9] and [10] coincide at $v \to 0$. Already for the second correction to the TF term this is not true as we shall see below. In the WK approach for $Z_2$ using (I-12) and (6) we have

$$Z_2(t) = \frac{1}{(2\pi\hbar)^3} \int \mathrm{d}p_x \, \mathrm{d}p_y \, \mathrm{d}p_z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, W_2(\vec{p}, \vec{x}; t) \, \mathrm{e}^{-Ht}, \tag{44}$$

with $W_2$ from (I-19). Integrating over the momenta, using the symmetry of the potential energy in (43), we have

$$Z_2(t) = \frac{t^{1/2}}{2(2\pi)^{3/2}\hbar} \left[ -g^2 I_1 + \frac{g^4 t}{4} I_2 \right], \tag{45}$$

where integrals $I_1$ and $I_2$ over $x$, $y$, $z$ are given by

$$I_1 = \int_0^\infty \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, x^2 \, \mathrm{e}^{-V(x,y,z)t}, \tag{46}$$

$$I_2 = \int_0^\infty \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, x^2 (y^2 + z^2) \, \mathrm{e}^{-V(x,y,z)t} \tag{47}$$

with

$$V(x, y, z) = \frac{g^2}{2}(x^2 y^2 + y^2 z^2 + z^2 x^2) + \frac{v^2}{2}(x^2 + y^2 + z^2). \tag{48}$$

Integrating over $x$ and using polar coordinates $r$, $\phi$ for the integrations over $y$ and $z$ we obtain

$$I_1 = \frac{1}{2} \left( \frac{2\pi}{t} \right)^{3/2} \int_0^\infty r \, \mathrm{d}r \, \frac{\exp\left(-\frac{1}{16} t g^2 r^4 - \frac{1}{2} t v^2 r^2\right) I_0\left(\frac{1}{16} t g^2 r^4\right)}{(v^2 + g^2 r^2)^{3/2}}, \tag{49}$$

$$I_2 = \frac{1}{2} \left( \frac{2\pi}{t} \right)^{3/2} \int_0^\infty r^5 \, \mathrm{d}r \, \frac{\exp\left(-\frac{1}{16} t g^2 r^4 - \frac{1}{2} t v^2 r^2\right) I_0\left(\frac{1}{16} t g^2 r^4\right)}{(v^2 + g^2 r^2)^{3/2}}, \tag{50}$$

where $I_0(z)$ denotes the modified Bessel function of the first kind. We see that $I_1$ is divergent at $r = 0$ if $v = 0$ in contrast to the TF term.

Again, we obtain a smooth transition in the limit $v \to 0$ if we make the substitution used in section 5 before taking the limit $v \to 0$. As for the $n = 2$ model, here also the zero-point quantum fluctuations in the channels generate an effective Higgs potential $\frac{1}{4}\hbar^2 g^2 t(x^2 + y^2 + z^2)$ and render the integral $I_1$ convergent. After this substitution, introducing the new variable $u^2 = \frac{1}{16} t g^2 r^4$, we get

$$Z_2(t) = \frac{\sqrt{2} t^{3/4}}{\hbar g^{1/2}} \left[ -J_0(\lambda) + 4 J_2(\lambda) \right] \tag{51}$$

with $\lambda = g t^{3/2} \hbar^2 (\ll 1)$ and

$$J_b(\lambda) = \int_0^\infty u^b \, \mathrm{d}u \, \frac{\mathrm{e}^{-u^2 - \lambda u}}{(\lambda + 8u)^{3/2}}. \tag{52}$$

Expanding the exponential function in the small parameter $\lambda$, the integral (52) can be expressed as a finite sum of the generalized hypergeometric functions. Retaining the main terms we have

$$J_0(\lambda) \approx \frac{1}{4\sqrt{\lambda}}, \qquad J_2(\lambda) \approx \frac{\Gamma\left(\frac{3}{4}\right)}{32\sqrt{2}}. \tag{53}$$

Thus, the second-order correction $Z_2(t)$ for the $n = 3$ YM model becomes

$$Z_2(t) \approx L \left[ 1 - \frac{\Gamma\left(\frac{3}{4}\right)^3}{2^{5/4}\pi^{3/2}} (g^2\hbar^4 t^3)^{1/4} \right],$$  (54)

where

$$L = \tfrac{1}{2}\Gamma\left(\tfrac{1}{4}\right)^3 (2\pi^2 g^2\hbar^4 t^3)^{-3/4}$$  (55)

is the TF term found in [9, 10]. We see from (54) that the quantum corrections at the order $\hbar^2$ are parametrically enhanced due to the quantum fluctuations generated by the effective Higgs potential $\frac{1}{2}\hbar^2 g^2(x^2 + y^2 + z^2)t$.

## 7. Conclusions

We have shown in I and here that the richness of the classical YM mechanics with the $x^2 y^2$ potential translates into, and even gets amplified by, the quantum mechanical properties of the system. The YM quantum mechanics exhibits a confinement property, which strongly influences the quantum mechanical motion in the $x^2 y^2$ potential. At higher orders in $\hbar$ (up to $\hbar^8$) this results in the vanishing of the leading quantum corrections, when we correctly take into account this property for the motion in the hyperbolic channels. We convinced that this property survives at higher orders, although we did not explicitly demonstrate it.

Here we calculated the quantum corrections to the partition function by adding a Higgs term to the potential, and found that power-like singularities arise in the limit $v \to 0$. We associate these essentially classical singularities with the fact that the Wigner–Kirkwood expansion does not take into account the effect of the quantum fluctuations on the motion within the channels. When these fluctuations, which are dictated by the uncertainty relation and the hyperbolic form of the channels, are taken into account, the escape along the coordinate axes, which is classically allowed, is prohibited, and the singularities disappear. As a result, the Thomas–Fermi term for the partition function acquires a renormalization expressed in terms of an asymptotic series in the parameter $\lambda^2 = g^2\hbar^4 t^3$ within both approaches.

We hope that the lessons we elicited from the present study of the higher-order quantum corrections to the homogeneous limit of the Yang–Mills equations will be useful for an improved understanding of the internal dynamics of the Yang–Mills quantum field theory.

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